Exercises

Taylor Approximation

Exercise 1. Compute the following Taylor polynomials.

(a) Compute $T_{2,1}(x)$ (i.e. the second ($n = 2$) Taylor Polynomial in the point $x_0 = 1$) for the function $f(x) = \ln(x)$.

(b) For $f(x) = \sin(x)$ compute $T_{1,0}(x)$, $T_{2,0}(x)$, $T_{3,0}(x)$, and $T_{4,0}(x)$.

(c) Compute $T_{3,0}$, $T_{4,0}$, $T_{5,0}$ for $f(x) = 4x^4 - 3x^3 + x^2 - 10x + 42$.

(d) Compute $T_{2,0}$ for the function $f(x) = x \cdot e^x$.

Exercise 2. Recall that the error of the Taylor approximation is given by

$$|f(x) - T_{n,x_0}(x)| = |R_{n,x_0}(x, \xi)| = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1} \right|$$

for some ξ between x and x_0.

(a) Consider the function $f(x) = \cos(x)$.

(i) Compute the Taylor polynomial of third order in $x_0 = \pi$ (i.e. $T_{3,\pi}(x)$) of f.

(ii) Find an estimate for the error of the approximation of f with $T_{3,\pi}(x)$ for $x \in [0, 2\pi]$, i.e. show that there is some $c \in \mathbb{R}$ such that

$$|R_{3,\pi}(x, \xi)| \leq c$$

for all $x \in [0, 2\pi]$.
Hint: Use that $|\cos(\xi)| \leq 1$.

(b) Consider the function $g(x) = \frac{1}{1-x}$.

(i) Compute the Taylor polynomial of second order in $x_0 = 4$ (i.e. $T_{2,4}(x)$) of g.

(ii) Find an estimate for the error of the approximation of g with $T_{2,4}(x)$ for $x \in [3, 5]$, i.e. show that there is some $c \in \mathbb{R}$ such that

$$|R_{2,4}(x, \xi)| \leq c$$

for all $x \in [0, 2\pi]$.
Hint: Use that if $x \in [3, 5]$ then $\xi \in [3, 5]$.

Exercise 3. Consider the function $f(x) = \sqrt{1+x}$.

(a) Compute $T_{1,0}(x)$ and the remainder $R_{1,0}(x, \xi)$.

(b) Use the fact $f(x) = T_{1,0}(x) + R_{1,0}(x, \xi)$ to prove that the following inequality holds for all $x \geq 0$:

$$\sqrt{1+x} \leq 1 + \frac{x}{2}$$